Boundedness of Pluricanonical Maps of varieties of general type - I

(Following Hacon - McKernan)

All varieties are over $\mathbb{C}$.

X smooth projective variety of general type i.e. ω_X is big.

Notation: $\phi_r : X \dashrightarrow \mathbb{P}(H^0(X, \omega_X^{\otimes r}))$

Since ω_X is big, ϕ_r is birational for $r >> 0$.

Question: How large does r have to be? In particular, does r depend on X?

Notation: $r(X)$ denote the smallest r for which ϕ_r is birational.

Main Theorem: For any $n \in \mathbb{N}$, there is a number r_n such that if X is smooth proj. variety of general type, $\dim X = n$, then

ϕ_r is birational for $r \geqslant r_n$

Definitions:

1) Bounded family of varieties:

$\{X_i\}$ is bounded if there is a finite type map of varieties

$\mathcal{X} \xrightarrow{\pi} B$ such that each

X_i is isomorphic to some fiber of π.

2) Birationally bounded: if the same happens but each X_i is only birational to a fiber of π.

3) Volume: X integral proj. var D big divisor

$$\mathrm{vol}(D) = \limsup_{m \to \infty} \frac{n!\, h^0(X, mD)}{m^n}$$

$n = \dim X$

Properties: 1) D is nef then

$$\mathrm{vol}(D) = \underbrace{(D \cdot D \cdot \ldots \cdot D)}_{\dim X \text{ times}}$$

2) $\mathrm{Vol}(rD) = r^n \mathrm{vol}(D)$

3) $\mathrm{vol}(K_X)$ is bounded in a bounded family of varieties of general type.

(projective vars)

Lemma: $X \overset{\varphi}{\dashrightarrow} Y$ birational map

$U \subset X$ domain of definition of φ

L effective divisor on Y. Then the divisor L' on X extending $\varphi^* L|_U$ satisfies:

$$\mathrm{vol}(L') \geq \mathrm{vol}(L)$$

Proof:

$$\begin{tikzcd} & X' \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ X \arrow[rr, dashed, "\varphi"] & & Y \end{tikzcd}$$

φ_1, φ_2 birational

$$\varphi_1^* L' = \varphi_2^* L + E$$

$\uparrow$ effective divisor

Example: X smooth proj. var dim n

assume ω_X is big. Fix $m \geq 1$

L - ample line bundle on X

$s \in H^0(X, L^m)$ $\quad$ $D = div(s)$ smooth divisor

$X_0 = X \setminus D$

$\mathcal{L}|_{X_0}$ is a line bundle $\quad$ $\mathcal{L}^m|_{X_0}$ is trivial

Consider the m^{th} cyclic cover of X over D - Y_m.

$$Y_m \longrightarrow X$$

$$Y_{0,m} = Y_m \setminus \pi^{-1}(D)$$

$$\begin{tikzcd} Y_{0,m} \arrow[r] \arrow[d] & X_{0,m} \arrow[d] \\ Y_m \arrow[r, "\pi"] & X \end{tikzcd}$$

π is fully ramified at D

$$D_Y = (\pi^* D)_{red}.$$

$$m D_Y = \pi^* D$$

Local calculation $\Rightarrow$

$$\omega_Y = \pi^* \omega_X + (m-1) D_Y$$

$$\omega_Y = \pi^*\left(K_X + \frac{m-1}{m} D\right)$$

$$\sim \pi^*\left(\omega_X \otimes \mathcal{L}^{m-1}\right)$$

$Y_1, Y_2, Y_3, \ldots$

$$(\omega_{Y_m}) = \pi^*(\omega_X \otimes \mathcal{L}^{m-1})$$

$$\mathrm{vol}(\omega_{Y_m}) = m\, \mathrm{vol}(\omega_X \otimes \mathcal{L}^{m-1})$$

$$\mathrm{vol}(\omega_{Y_m}) \longrightarrow \infty \quad \text{as } m \to \infty$$

$\Longrightarrow$ $\{Y_m\}$ are not a bounded family

Motivation / Sketch of Proof:

$$\phi_r : X \dashrightarrow \mathbb{P}(H^0(X, \omega_X^r))$$

is birational, its enough to show ω_X^r separates very general points. i.e. there a countable union of closed subvarieties of X say $Z_1, Z_2, \ldots$ s.t. if $x, y \in X \setminus \bigcup_i Z_i$ then

$$H^0(X, \omega_X^r) \longrightarrow \mathbb{C}_x \oplus \mathbb{C}_y$$

is surjective.

Suppose
X is a curve of $g \geq 2$

$$H^1(X, rK_X - P - Q) = 0$$

$r \geq 3$ for any $P, Q \in X$.

So, we get surjectivity.

If $\dim X > 1$, we'll use Nadel vanishing in the following way:

Prop: Suppose there is a $\lambda > 0$ s.t. for any two very general points x, y there is a $\mathbb{Q}$-divisor $D_{x,y}$ s.t.

1) $D_{x,y} \underset{\mathbb{Q}}{\sim} \lambda K_X$

2) x is an isolated point of $\mathrm{Zeroes}(\mathcal{J}(D_{x,y}))$ and $y \in \mathrm{Zeroes}(\mathcal{J}(D_{x,y}))$

Then ϕ_r is birational if $r \geq \lambda + 1$

$r(X) \leq \lambda + 1$

Proof: ω_X is big, we choose $m > 0$ and an ample H s.t.

$$mK_X = H + G$$

$G \geq 0$ effective

G does not pass through x & y.

$$D'_{x,y} = D_{x,y} + \frac{r - (1+\lambda)}{m} G$$

$$(r-1)K_X - D'_{x,y} \underset{\mathbb{Q}}{\sim} \frac{(r-1)(G+H)}{m} - \frac{\lambda}{m}(G+H)$$

$$- \left(\frac{r-(1+\lambda)}{m}\right) G$$

$$\sim_{\mathbb{Q}} \frac{(r-1-\lambda)}{m} H \quad \text{ample}$$

Nadel vanishing $\Rightarrow H^i(X, \omega_X^{r} \otimes \mathcal{J}(D'_{x,y}))$

$= 0$ for $i > 0$

$D'_{x,y}$ also satisfies condition (2).

$$0 \longrightarrow \omega_X^{\otimes r} \otimes \mathcal{J}(D'_{x,y}) \longrightarrow \omega_X^{\otimes r}$$

$$\longrightarrow \omega_X^{\otimes r} \otimes \mathcal{O}_X / \mathcal{J}(D'_{x,y}) \to 0$$

Nadel vanishing $\Longrightarrow$

$$H^0(\omega_X^{r}) \twoheadrightarrow H^0(\omega_X^{\otimes r} \otimes \mathcal{O}_X / \mathcal{J}(D'_{x,y}))$$

Since x was an isolated point of

zeroes of $\mathcal{J}(D'_{x,y})$, we can choose

$s \in H^0(\omega_X^{\gamma})$ that vanishes at y but not at x.

By symmetry, we get the surjectivity. $\boxtimes$

Main challenge: bound λ for which we can produce $D_{x,y}$. (want $\lambda \leq m(n)$)

Something slightly weaker is sufficient:

Prop: Suppose $r(X) \leq \frac{A}{\mathrm{vol}(\omega_X)^{1/n}} + B$

for A, B constants

Then there is an r_n s.t.

$$r(X) \leq r_n \text{ for all } X \text{ of general type.}$$

Pf: $vol(K_X) \geq 1$, then

$r(X) \leq A + B$.

$vol(K_X) < 1$, then

let $\phi_{r(X)} : X \dashrightarrow \mathbb{P}(H^0(X, \omega_X^r))$

Z closure of $\phi(X)$

$$\deg(Z) \leq vol(\omega_X^r) = r^n vol(\omega_X)$$
$$\leq \left(\frac{A}{vol(K_X)^{1/n}} + B + 1\right)^n \cdot vol(K_X)$$
$$\leq (A + B + 1)^n$$

Hypothesis $\Rightarrow$ if X is of general type with $vol(X) \leq 1$, then X is birational to a bounded degree subvariety of some projective space.

$\Longrightarrow$ Such varieties form a bounded family.

Lemma: $\pi: X \longrightarrow B$ be a bounded family of proj. varieties of general type. Then there is an $R > 0$ s.t. if Y is a resolution of any fiber of π, then $r(Y) \leq R$

Pf: We consider a resolution of fiber over each generic point of B and use Noetherian induction. $\boxtimes$

Log Canonical Centers:

(X, Δ) pair i.e. X - normal variety

$\mathbb{Q}$-div Δ s.t. $K_X + \Delta$ is $\mathbb{Q}$-Cartier

$\mu : Y \longrightarrow X$ log-resolution of (X, Δ)

i.e. Y smooth, μ - proper

$\mu^*(\Delta) \cup Exc(\mu)$ has snc support.

Write $K_Y + \Gamma = \mu^*(K_X + \Delta)$

$$\Gamma = \sum a_i \Gamma_i \qquad \Gamma_i \text{ -prime divisors}$$

Log discrepancy of (X, Δ) w.r.t.

$\Gamma_i = 1 - a_i$

A log-canonical center is an irreducible subvar. $V \subset X$ s.t. V is the image of $\mu(E)$ for some E (prime divisor) s.t. $1 - a(E) \leq 0$

A log canonical place is a valuation corresponding to E as above.

$LLC(X, \Delta, x)$ - the set of all log-canonical centers containing $x \in X$.

Main Lemma:

(X, Δ) pair $\quad x \in X$ closed point

x klt point of X and

(X, Δ) is log canonical near x.

If $W_1, W_2 \in LLC(X, \Delta, x)$ and

W is irreducible component of $W_1 \cap W_2$

then $W \in LLC(X, \Delta, x)$

If (X, Δ) is not klt, then $LLC(X, \Delta, x)$

has a unique minimal irred. element

V.

Moreover, there is a $\mathbb{Q}$-divisor E s.t.

$$LLC(X, (1-\varepsilon)\Delta + \varepsilon E, x) = \{V\} \quad \text{for } 0 < \varepsilon \ll 1$$

Lemma: let (X, Δ) log pair. x smooth point of X. If $\mathrm{mult}_x(\Delta) \geq \dim X$ then $LLC(X, \Delta, x) \neq \emptyset$.

If $\mathrm{mult}_x(\Delta) < 1$, then

$$LLC(X, \Delta, x) = \emptyset.$$